Solution Set 7 (compiled by Daniel Larson)

1. **Griffiths 6.17** In a linear material, we know H is proportional to B: $\mathbf{B} = \mu\mathbf{H} = \mu_0(1 + \chi_m)\mathbf{H}$, so for a long wire it should be circumferential. We can then use Ampere's law to find $\mathbf{H}$ from the free current, and then get B from H. As usual, we draw an amperian loop around the wire:

$$\oint \mathbf{H} \cdot d\mathbf{l} = 2\pi s H(s) = I_{f_{\text{enc}}} = \begin{cases} I(s^2/a^2), & (s < a) \\ I & (s > a) \end{cases}$$

$$H(s) = \begin{cases} \frac{Is}{2\pi a^2}, & (s < a) \\ \frac{I}{2\pi s}, & (s > a) \end{cases} \Rightarrow B(s) = \begin{cases} \frac{\mu_0(1+\chi_m)Is}{2\pi a^2}, & (s < a) \\ \frac{\mu_0 I}{2\pi s}, & (s > a) \end{cases}.$$

In a linear material, $J_b = \chi_m J_f = \chi_m \frac{I}{\pi a^2}$ (using the fact that I is uniform over the area of the wire) and points in the same direction as I. $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = \chi_m \mathbf{H}(a) \times \hat{\mathbf{n}} \Rightarrow K_b = \frac{\chi_m I}{2\pi a}$ in the direction *opposite* from I (using the right-hand-rule). The total bound current is $I_b = \pi a^2 J_b + 2\pi a K_b = \chi_m I - \chi_m I = 0$ as it must be.

2. **Griffiths 6.21**

   (a) We need to compute the work it takes to bring the magnetic dipole in from infinity to the origin and rotate it to its final configuration. First, bring the dipole to the origin along a trajectory in which $\mathbf{m}$ is always perpendicular to $\mathbf{B}$ so that there is no force on the dipole and hence no work done. For simplicity, imagine $\mathbf{B}$ is uniform and points in the $\hat{\mathbf{y}}$ direction. Then we can slide a dipole (pointing in the $\hat{\mathbf{x}}$ direction) in along the x-axis. All the work comes from rotating the dipole in the presence of the B-field. The torque exerted by the B-field is $\mathbf{N} = \mathbf{m} \times \mathbf{B} = mB\sin\theta\,\hat{\mathbf{z}}$ where θ is the angle between $\mathbf{m}$ and $\mathbf{B}$ (initially $\pi/2$); this is opposite the torque we must exert in order to rotate the dipole. So to move the dipole from an angle of $\pi/2$ with respect to $\mathbf{B}$ to some other angle θ we must do an amount of work $U = \int_{\pi/2}^{\theta} mB\sin\theta'\,d\theta' = mB(-\cos\theta')|_{\pi/2}^{\theta} = -mB\cos\theta = -\mathbf{m} \cdot \mathbf{B}$.

   (b) We can put the first diple at the origin. It produces a magnetic field $\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1]$ at any location $\mathbf{r}$. The second dipole, located at $\mathbf{r}$, interacts with this magnetic field as in part (a). Thus $U = -\mathbf{m}_2 \cdot \mathbf{B}_1 = -\frac{\mu_0}{4\pi r^3}[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\mathbf{m}_2 \cdot \hat{\mathbf{r}} - \mathbf{m}_2 \cdot \mathbf{m}_1] = \frac{\mu_0}{4\pi r^3}[\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})]$.

   (c) From the figure, $\mathbf{m}_i \cdot \hat{\mathbf{r}} = m_i \cos\theta_i$ for $i = 1$ or 2, and $\mathbf{m}_1 \cdot \mathbf{m}_2 = m_1 m_2 \cos(\theta_1 - \theta_2) = m_1 m_2 (\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$. So $U = \frac{\mu_0 m_1 m_2}{4\pi r^3}[\cos(\theta_1 - \theta_2) - 3\cos\theta_1\cos\theta_2] = \frac{\mu_0 m_1 m_2}{4\pi r^3}[\sin\theta_1\sin\theta_2 - 2\cos\theta_1\cos\theta_2]$. A stable configuration occurs when the energy is at a minimum.

   $$\frac{\partial U}{\partial\theta_1} = \frac{\mu_0 m_1 m_2}{4\pi r^3}(\cos\theta_1\sin\theta_2 + 2\sin\theta_1\cos\theta_2) = 0 \quad \Rightarrow \quad 2\sin\theta_1\cos\theta_2 = -\cos\theta_1\sin\theta_2$$

   $$\frac{\partial U}{\partial\theta_2} = \frac{\mu_0 m_1 m_2}{4\pi r^3}(\sin\theta_1\cos\theta_2 + 2\cos\theta_1\sin\theta_2) = 0 \quad \Rightarrow \quad 2\sin\theta_1\cos\theta_2 = -4\cos\theta_1\sin\theta_2$$

   So we need $\cos\theta_1\sin\theta_2 = \sin\theta_1\cos\theta_2 = 0$. This will happen for either $\sin\theta_1 = \sin\theta_2 = 0 \Rightarrow$ (i) $\rightarrow\rightarrow$ or (ii) $\rightarrow\leftarrow$; or if $\cos\theta_1 = \cos\theta_2 = 0 \Rightarrow$ (iii) $\uparrow\uparrow$ or (iv) $\uparrow\downarrow$. We know that the lowest energy configuration will have $\mathbf{m}$ lined up with $\mathbf{B}$. This only happens in (i) and (iv), so they are the stable minima. To find the absolute minimum, we need to calculate U. For situation (i) we have $\theta_1 = \theta_2 = 0$ so $U = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-2)$ whereas for (iv) we have $\theta_1 = -\theta_2 = \pi/2$, so $U = \frac{\mu_0 m_1 m_2}{4\pi r^3}(-1)$. Thus the most stable configuration is the one with the lowest energy, namely (i) where the magnetic moments are lined up along the line joining them: $\rightarrow\rightarrow$.

   (d) Using the result from part (c), the most stable configuration should be when the dipoles all form one line, pointing in one direction: $\rightarrow\rightarrow\rightarrow\rightarrow\rightarrow$.

3. **Griffiths 6.26** The angle θ_1 is related to the components of B_1 which are parallel and perpendicular to the interface: $\tan\theta_1 = \frac{B_1^\parallel}{B_1^\perp}$. The same relation holds for θ_2 and B_2. The perpendicular components of B are continuous across the boundary, so $B_1^\perp = B_2^\perp$. We also know that the parallel components of H are continuous across the boundary, since there is no free surface current. Since $\mathbf{B} = \mu\mathbf{H}$ this gives: $H_1^\parallel = H_2^\parallel \Rightarrow \frac{1}{\mu_1}B_1^\parallel = \frac{1}{mu_2}B_2^\parallel$. Putting these together:

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{B_2^\parallel}{B_2^\perp}\frac{B_1^\perp}{B_1^\parallel} = \frac{B_2^\parallel}{B_1^\parallel} = \frac{\mu_2}{\mu_1}$$

4. **Griffiths 7.3**

(a) To find the resistance, we need to look at the ration of the potential difference to the current flowing between to metal objects. Any currents flowing will leave conductor 1 and flow to conductor 2. So we can find the current by enclosing conductor 1 with a surface and then evaluating $I = \int \mathbf{J} \cdot d\mathbf{a}$. This equation is exactly what we need. First, Gauss's law tells us $\int \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\epsilon_0}Q_{\text{enc}}$, while Ohm's law gives $\mathbf{J} = \sigma\mathbf{E}$ and $V = IR$. We assume there are no free charges floating around in our conducting material, so Q_{enc} is simply the charge on the first object, which is related to the capacitance of the system by $Q = CV$. These are all the ingredients we need.

$$I = \int \mathbf{J} \cdot d\mathbf{a} = \sigma \int \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma}{\epsilon_0}Q = \frac{\sigma}{\epsilon_0}CV = \frac{\sigma}{\epsilon_0}CIR \Rightarrow R = \frac{\epsilon_0}{\sigma C}.$$

(b) We apply a potential difference V_0 between objects 1 and 2 and then allow the charge to leak off. The voltage at any time is given by $V(t) = I(t)R = -\frac{dQ}{dt}R$, where the minus sign comes because we assume the current I is positive, but we know the charge Q is decreasing. We also know that $V = Q/C$, so that tells us $\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt}$, because capacitance is just a constant. Thus $V(t) = -RC\frac{dV}{dt} \Rightarrow \frac{dV}{dt} = -\frac{1}{RC}V(t) \Rightarrow V(t) = V(0)e^{-t/RC} = V_0e^{-t/RC}$. Then the time constant $\tau = RC = \epsilon_0/\sigma$.

5. **Griffiths 7.7**

(a) Current will flow due to the changing flux in the loop formed by the bar and the wire. The total flux through the loop is $\Phi = BA$. If the bar is moving at speed v to the right, the area is changing at a rate of $\frac{dA}{dt} = lv$. Thus $\mathcal{E} = -\frac{d\Phi}{dt} = -Blv$. Then $\mathcal{E} = IR \Rightarrow I = Blv/R$. The minus sign just refers to the direction, but it is easier to figure that out using Lenz's law. Since the flux into the page is increasing, the current will flow to produce flux coming out of the page, so the current will be going *down* through the resistor.

(b) There is magnetic force on the bar because there is a current flowing in the presence of a magnetic field. $F = \int Id\mathbf{l} \times \mathbf{B} = IlB = B^2l^2v/R$ and it points to the left, which is the direction of $d\mathbf{l} \times \mathbf{B}$.

(c) The force on the bar is slowing it down so we take it to be negative.

$$F = -\frac{1}{R}B^2l^2v = ma = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = -\frac{B^2l^2}{Rm}v \Rightarrow v(t) = v_0e^{-B^2l^2t/Rm}.$$

(d) The energy goes into heading the resisitor. The power delivered to the resisitor is

$$P = \frac{dW}{dt} = I^2R = \frac{B^2l^2}{R}v_0^2e^{-2\alpha t}, \text{ where } \alpha = \frac{B^2l^2}{Rm}; \Rightarrow \frac{dW}{dt} = \alpha mv_0^2e^{-2\alpha t}.$$

The bar keeps slowing down, but takes an infinite amount of time to stop. During this time, the total energy delivered to the resistor is

$$W = \alpha mv_0^2 \int_0^\infty e^{-2\alpha t}\,dt = \alpha mv_0^2 \left.\frac{e^{-2\alpha t}}{-2\alpha}\right|_0^\infty = \alpha mv_0^2\frac{1}{2\alpha} = \frac{1}{2}mv_0^2.$$

6. **Griffiths 7.11** Let l be the width of the loop, and s be the distance between the top edge of the loop and the bottom of the region of B-field. The flux through the loop is $\Phi = Bla$, so $\mathcal{E} = -\frac{d\Phi}{dt} = -Bl\frac{ds}{dt}$. Let's only consider magnitudes and drop the minus sign. Since $\frac{ds}{dt} = v(t)$, the velocity of the loop at time t, we have $\mathcal{E} = Blv = IR$, assuming the loop has resistance R. Then $I = Blv/R$ is the current flowing in the loop. As the loop falls, the flux into the page is decreasing, so the current flows in a clockwise direction to oppose the change in flux. But the part of the loop still in the region of magnetic field will feel a force because there is a current in a magnetic field. The forces on the two sides will cancel, leaving an upward force of magnitude $F = IlB = B^2l^2v/R$. This force opposes the force of gravity, $F_g = mg$ which pulls the loop downward. The loop will have reached terminal velocity, v_t, when these two forces balance: $mg = B^2l^2v_t/R \Rightarrow v_t = (mgR)/(B^2l^2)$.

To find the velocity as a function of time, we need Newton's second law: $F_{\text{net}} = ma = m\frac{dv}{dt} = mg - \frac{B^2l^2}{R}v$ where I have taken the downward direction to be positive. Letting $\alpha = B^2l^2/mR$, we have $v_t = g/\alpha$, and we get a differential equation for the velocity:

$$\frac{dv}{dt} = g - \alpha v \Rightarrow \frac{dv}{g - \alpha v} = dt \Rightarrow -\frac{1}{\alpha}\ln(g - \alpha v) = t + \text{const.} \Rightarrow g - \alpha v = Ae^{-\alpha t}$$

Since the loop starts at rest at $t = 0$, the constant $A = g$. Thus $v(t) = \frac{g}{\alpha}(1 - e^{-\alpha t}) = v_t(1 - e^{-\alpha t})$. At 90% of terminal velocity we have $v/v_t = 0.9 = 1 - e^{-\alpha t} \Rightarrow e^{-\alpha t} = 0.1 \Rightarrow t = \frac{1}{\alpha}\ln 10 = \frac{v_t}{g}\ln 10$.

To get a numerical answer, we need various properties of aluminum and the dimensions of the loop. Assume the loop is square, with sides l and cross-sectional area A. The resistivity is $\rho = \frac{1}{\sigma} = 2.65 \times 10^{-8}$ Ω m; the mass density is $\eta = 2.7 \times 10^3$ kg/m^3; $g = 9.8$ m/s^2; and $B = 1$ T. The resistance of a piece of metal with uniform cross-sectional area A and length L is $R = \frac{L}{A\sigma}$, so in this case we have $R = \frac{4l\rho}{A}$.

$$v_t = \frac{mgR}{B^2l^2} = \frac{(\eta A4l)g(4l\rho/A)}{B^2l^2} = \frac{16\eta g\rho}{B^2} = 1.1 \text{ cm/s}; \Rightarrow t_{90\%} = \frac{v_t}{g}\ln 10 = 2.8 \text{ ms}$$

Finally, if the loop were cut, no current would flow, so there wouldn't be any force to oppose gravity and the loop would fall freely under the force of gravity.

7. **Griffiths 7.17**

   (a) We assume that the solenoid is relatively long, so the only magnetic field in the loop is the uniform B-field inside the solenoid, namely $B = \mu_0 nI$. Thus the flux passing through the loop is $\Phi = \pi a^2 B = \pi a^2 \mu_0 nI \Rightarrow \mathcal{E} = -\pi a^2 \mu_0 n\frac{dI}{dt}$. The negative sign just refers to the direction, which is easier to find using Lenz's law, so we'll ignore it. The magnitude of the current passing through the resistor is given by $\mathcal{E} = I_r R \Rightarrow I_r = \frac{1}{R}\pi a^2 \mu_0 nk$. The flux due to the solenoid is pointing to the right and is increasing, thus the current in the loop will flow in order to produce a flux inside the loop pointing to the left, which is opposite the direction of the current flowing in the solenoid, or to the right in the picture in the text.

   (b) When the solenoid is pulled out and reinserted there is lots of changes going on in the flux, most of them very complicated. But all we need to know to get the total charge is the total change in flux.

$$\Delta Q = \int I\, dt = \int \frac{\mathcal{E}}{R} = \int -\frac{1}{R}\frac{d\Phi}{dt} = -\frac{1}{R}(\Phi_f - \Phi_i) \Rightarrow \Delta Q = \frac{1}{R}\Delta\Phi \quad \text{(in magnitude)}$$

   Initially there is flux $\Phi_i = \pi a^2 \mu_0 nI$ pointing to the right, and at the end there is the same amount of flux pointing in the opposite direction, the net change in flux is $\Delta\Phi = 2\pi a^2 \mu_0 nI$, which means $\Delta Q = \frac{1}{R}2\pi a^2 \mu_0 nI$.

8. **Griffiths 7.48** Starting with Equation (5.3), we have $qBR = mv$. Keeping R fixed, we can differentiate with respect to time: $qR\frac{dB}{dt} = m\frac{dv}{dt} = ma = F = qE$. Thus $E = R\frac{dB}{dt}$, where B is evaluated at the radius of the electron's orbit, R. From Faraday's law we know $\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}$, so if we take the loop to be the electron's orbit at radius R, $2\pi RE = -\frac{d\Phi}{dt}$. Combining this with the previous result we can solve for B:

$\frac{dB}{dt} = -\frac{1}{2\pi R^2}\frac{d\Phi}{dt} \quad \Rightarrow \quad B = -\frac{1}{2}\left(\frac{\Phi}{\pi R^2}\right) + C$ where C is some integration constant. If $B = 0$ when $t = 0$, there will be no flux through the loop, so the constant must be zero. But this means $B(R) = -\frac{1}{2}\left(\frac{\Phi}{\pi R^2}\right)$. The term in parentheses is simply the total field throughout the orbit (flux) divided by the area of the orbit, namely the average field. Thus the average field over the orbit is twice the value of the field at the circumference (in magnitude).